Stability of the essential spectrum for $2D$–transport models with Maxwell boundary conditions.

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Abstract

We discuss the spectral properties of collisional semigroups associated to various models from transport theory by exploiting the links between the so-called resolvent approach and the semigroup approach. Precisely, we show that the essential spectrum of the full transport semigroup coincides with that of the collisionless transport semigroup in any L^p–spaces $(1 < p < \infty)$ for three $2D$–transport models with Maxwell–boundary conditions.

Keywords: Transport theory, essential spectrum, perturbed semigroup, boundary conditions.

1 Introduction

This work follows the very recent one of the first author on several mono–energetic transport problems [18] by dealing now with collisional models. Precisely, we show that the essential spectrum of the full transport semigroup coincides with that of the collisionless transport semigroup associated to the following $2D$–models:

i) The Rotenberg model with boundary conditions of Maxwell type.

ii) The one–velocity transport equation in a sphere with Maxwell–type boundary conditions.

iii) The mono–energetic transport equation in a slab of thickness $2a > 0$.

These three models are particular versions of the more general transport equation

$$\frac{\partial \phi}{\partial t}(x,\xi,t) + \xi \cdot \nabla_x \phi(x,\xi,t) + \sigma(x,\xi)\phi(x,\xi,t) = \int_V \kappa(x,\xi,\xi_\star)\phi(x,\xi_\star,t)\mathrm{d}\nu(\xi_\star), \tag{1.1}$$

with the initial condition

$$\phi(x,\xi,0)=\phi_0(x,\xi) \qquad\qquad (x,\xi)\in\Omega\times V \tag{1.2}$$

and with Maxwell boundary conditions

$$\phi_{|\Gamma_-}(x,\xi,t)=H(\phi_{|\Gamma_+})(x,\xi,t) \qquad\qquad (x,\xi)\in\Gamma_-, t>0 \tag{1.3}$$

where Ω is a smooth open subset of $\mathbb{R}^N$ $(N\geqslant 1)$, V is the support of a positive Radon measure $\mathrm{d}\nu$ on $\mathbb{R}^N$ and $\phi_0\in X_p:=L^p(\Omega\times V,\mathrm{d}x\mathrm{d}\nu(\xi))$ $(1\leqslant p<\infty)$. Here Γ_- (resp Γ_+) denotes the incoming (resp. outgoing) part of the boundary of the phase space $\Omega\times V$, $\Gamma_\pm=\{(x,\xi)\ \in\partial\Omega\times V\ ;\ \pm\xi\cdot n(x)>0\}$ where $n(x)$ stands for the outward normal unit at $x\in\partial\Omega$. The boundary condition (1.3) expresses that the incoming flux $\phi_{|\Gamma_-}(\cdot,\cdot,t)$ is related to the outgoing one $\phi_{|\Gamma_+}(\cdot,\cdot,t)$ through a *linear operator* H that we shall assume to be bounded on some suitable trace spaces. The collision operator $\mathcal{K}$ arising at the right-hand–side of (1.1) is assumed to be a bounded operator in $L^p(\Omega\times V,\mathrm{d}x\mathrm{d}\nu(\xi))$ $(1\leqslant p<\infty)$ and it is well–known that $\mathcal{K}$ induces some compactness with respect to the velocity ξ. The well–posedness of the free–streaming version of (1.1) (corresponding to null collision $\kappa=0$) has been investigated recently in [16, 19] where sufficient conditions on the boundary operator H are given ensuring that the free streaming operator generates a c_0–semigroup $(U(t))_{t\geqslant 0}$ in $L^p(\Omega\times V,\mathrm{d}x\mathrm{d}\nu(\xi))$. Then, since $\mathcal{K}$ acts as a *bounded perturbation* of $(U(t))_{t\geqslant 0}$, the model (1.1)–(1.3) is governed by a c_0–semigroup $(V(t))_{t\geqslant 0}$ in $L^p(\Omega\times V,\mathrm{d}x\mathrm{d}\nu(\xi))$.

It is well-known that the asymptotic behavior (as $t\to\infty$) of the solution $\phi(\cdot,\cdot,t)$ to (1.1)–(1.3) is strongly related to the spectral properties of the semigroup $(V(t))_{t\geqslant 0}$. In particular, an important task is the stability of the essential spectrum [29]: does

$$\sigma_{\mathrm{ess}}(V(t))=\sigma_{\mathrm{ess}}(U(t)) \tag{1.4}$$

for any $t\geqslant 0$?

This question has been answered positively in the case of non–reentry boundary conditions (i.e. $H=0$) in [23, 28] by showing that the difference $V(t)-U(t)$ is a compact operator in $L^p(\Omega\times V,\mathrm{d}x\mathrm{d}\nu(\xi))$ $(1<p<\infty)$. The case of re–entry boundary conditions is much more involved because of the difficulty to compute the semigroup $(U(t))_{t\geqslant 0}$ in this case. There exists a few partial results dealing with the above models i)–iii) [4, 5, 6, 7, 31, 10, 13, 34, 35] but the asymptotic behavior of the solution to the associated equations is investigated only for smooth initial data or, at best, estimates of the essential type of the $(V(t))_{t\geqslant 0}$ are provided. Therefore, question (1.4) is *totally open* for the aforementioned models.

The present paper generalizes the previous ones by establishing the above identity (1.4) for the three above models i)-iii) in the case $1<p<\infty$. The strategy is based upon the so–called resolvent approach, already used in [15, 21, 13], and exploits the link between this approach and the compactness of $V(t)-U(t)$ recently discovered, in [2, 20, 28] (see Section 2 for more details). Note that the results of [2, 28] are valid only *in a Hilbert space setting* but we will see in this paper how they allow to treat the above three above models in any L^p–space with $1<p<\infty$. Indeed, for *Maxwell-like boundary conditions* in $2D$–geometry, the boundary operator H splits as $H=\mathcal{J}+K$ where $\mathcal{J}$ is a multiplication operator and K is compact. This allows to approximate it by some finite–rank operators. Under some natural assumptions on the collision operator $\mathcal{K}$, it is then possible to approximate both $U(t)$ and $V(t)$ in such a way that both of them are bounded operator in any $L^r(\Omega\times V,\mathrm{d}x\mathrm{d}\nu(\xi))$, $1<r<\infty$. Then, by an interpolation argument, it is sufficient to prove the compactness of the difference $V(t)-U(t)$ in the Hilbert space $L^2(\Omega\times V)$. This strategy excludes naturally the case $p=1$ for which a specific analysis is necessary.

Let us explain more in details the content of the paper. In the following section, we present the resolvent approach and the result of the second author we shall use in the rest of the paper. In section 3, we investigate

the Rotenberg model and gave a precise description of the method of the proof of identity (1.4). In section 4, we deal with the mono–energetic transport equation in a sphere by adopting the approach exposed in Section 3. Finally, we deal in section 5 with the model iii).

Notations. Given two Banach spaces X and Y, $\mathfrak{B}(X,Y)$ shall denote the set of bounded linear operators from X to Y whereas the ideal of compact operators from X to Y will be denoted $\mathfrak{C}(X,Y)$. When $X=Y$, we will simply write $\mathfrak{B}(X)$ and $\mathfrak{C}(X)$.

2 On the resolvent approach

We recall here the link between the so–called resolvent approach and the study of the compactness of the difference of semigroups. Let X be a Banach space and let $T:\mathrm{Dom}(T)\subset X\to X$ be the infinitesimal generator of a c_0-semigroup of operators $(U(t))_{t\geqslant 0}$ in X. We consider the Cauchy problem

$$\begin{cases}\dfrac{\mathrm{d}\phi}{\mathrm{d}t}(t) & =(T+K)\,\phi(t) \qquad t\geqslant 0,\\ \phi(0) & =\phi_0\end{cases} \tag{2.1}$$

where $K\in\mathfrak{B}(X)$ and $\phi_0\in X$. Since $A:=T+K$ is a bounded perturbation of T, it is known that A with domain $\mathrm{Dom}(A)=\mathrm{Dom}(T)$ generates a c_0-semigroup $(V(t))_{t\geqslant 0}$ on X given by the Dyson–Phillips expansion

$$V(t)=\sum_{j=0}^{\infty}U_j(t) \tag{2.2}$$

where $U_0(t)=U(t)$, $U_j(t)=\displaystyle\int_0^t U(t-s)KU_{j-1}(s)ds$ $(j\geqslant 1)$.

When dealing with the time–asymptotic behavior of the solution $\phi(t)$ to (2.1), until recently, two techniques have been used. The first one, called the *semigroup approach*, consists in studying the remainder $R_n(t)=\sum_{j\geqslant n}U_j(t)$ of the Dyson–Phillips expansion (2.2) (see [33]). Actually, if there is $n\geqslant 0$ such that $R_n(t)\in\mathfrak{C}(X)$ for any $t\geqslant 0$ then $\sigma(V(t))\cap\{\mu\in\mathbb{C}\,;\,|\mu|>\exp(\eta t)\}$ consists of, at most, isolated eigenvalues with finite algebraic multiplicities where η is the type of $(U(t))_{t\geqslant 0}$. Therefore, for any $\nu>\eta$, $\sigma(T)\cap\{\mathrm{Re}\lambda>\nu\}$ consists of a finite set of isolated eigenvalues $\{\lambda_1,\ldots,\lambda_n\}$. Then, the solution to (2.1) satisfies

$$\lim_{t\to\infty}\exp(-\beta t)\|\phi(t)-\sum_{j=1}^{n}\exp(\lambda_j t+D_j t)P_j\phi_0\|=0 \tag{2.3}$$

where $\phi_0\in X$, P_j and D_j denote, respectively, the spectral projection and the nilpotent operator associated to $\{\lambda_i,\ i=1,\ldots,n\}$ and

$$\sup\{\mathrm{Re}\lambda,\ \lambda\in\sigma(T_H),\ \mathrm{Re}\lambda<\nu\}<\beta<\min\{\mathrm{Re}\lambda_j,\ j=1,\ldots,n\}.$$

Of course, the success of such a method is strongly related to the possibility of computing the terms of the Dyson–Phillips expansion (2.2). Until recently, it appeared to be the only way to discuss their compactness properties. Unfortunately, in practical situations, the unperturbed semigroup $(U(t))_{t\geqslant 0}$ may not be explicit or at least can turn out to be hard to handle.

An alternative way to determine the long–time behavior of $\phi(t)$ is the so–called *resolvent approach* initiated by J. Lehner and M. Wing [15] in the context of neutron transport theory and consists in expressing

$\phi(t)$ as an inverse Laplace transform of $(\lambda - T - K)^{-1}\phi_0$. This method has been developed subsequently in an abstract setting by M. Mokhtar–Kharroubi [21] and, more recently by Degong Song [30] (see [13] for an application of the results of [30] in the context of neutron transport equation on a slab). The main drawback of this approach is that (2.3) is valid only for smooth initial data $\phi_0 \in D(A)$. In particular, even in Hilbert spaces, it does not permit to explicit the essential type of $V(t)$ but only to give some estimates of it [30].

In a Hilbert space setting, these two approaches have been linked recently by S. Brendle [2]. Precisely, if there exist some $\alpha > w_0(U)$ and some integer m such that

$$(\lambda - T)^{-1}\left(K(\lambda - T)^{-1}\right)^m \qquad \text{is compact for any } \mathrm{Re}\lambda = \alpha$$

and

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|(\lambda - T)^{-1}\left(K(\lambda - T)^{-1}\right)^m\| = 0 \qquad \forall \mathrm{Re}\lambda = \alpha$$

then the $(m+2)$–remainder term $R_{m+2}(t)$ of the Dyson–Phillips expansion series is compact. Such a result, though really important for the applications, does not allow to investigate the compactness of the difference of the two semigroups $V(t) - U(t) = R_1(t)$. Very recently, the second author, inspired by the work of S. Brendle [2], has been able to provide sufficient conditions in terms of the resolvent of T ensuring the compactness of the *first remainder term* $R_1(t)$. Precisely [28, Corollary 2.2, Lemma 2.3],

Theorem 2.1 *Assume that T is dissipative and there exists $\alpha > w_0(U)$ such that*

$$(\alpha + i\beta - T)^{-1}K(\alpha + i\beta - T)^{-1} \qquad \textit{is compact for all } \beta \in \mathbb{R} \tag{2.4}$$

and

$$\lim_{|\beta|\to\infty} \left(\|K^\star(\alpha + i\beta - T)^{-1}K\| + \|K(\alpha + i\beta - T)^{-1}K^\star\|\right) = 0 \tag{2.5}$$

then $R_1(t)$ is compact for all $t \geqslant 0$. In particular, $\sigma_{\mathrm{ess}}(V(t)) = \sigma_{\mathrm{ess}}(U(t))$.

We refer to [28] for a proof of this result as well as for its application to neutron transport equation in bounded geometry with absorbing boundary conditions.

Remark 2.2 *Actually, under the hypothesis* (2.5)*, the mapping $t \geqslant 0 \mapsto R_1(t) \in \mathfrak{B}(X)$ is continuous [28]. This implies the stability of the critical spectrum (see [25] for a precise definition) $\sigma_{\mathrm{crit}}(V(t)) = \sigma_{\mathrm{crit}}(U(t))$ for any $t \geqslant 0$. Such an identity plays a crucial role for establishing spectral mapping theorems (see [24] for a recent application to neutron transport equations in unbounded geometries).*

Remark 2.3 *Note that Assumption* (2.4) *implies that $(\lambda - T - K)^{-1} - (\lambda - T)^{-1} \in \mathfrak{C}(X)$ for any $\lambda \in \rho(T + K)$. Therefore, $\sigma_{\mathrm{ess}}(T + K) = \sigma_{\mathrm{ess}}(T)$.*

Let us recall now the definition of regular collision operators as they appear in [22]. The notations are those of the introduction.

Definition 2.4 *An operator $\mathcal{K} \in \mathfrak{B}(X_p)$ $(1 < p < \infty)$ is said to be regular if $\mathcal{K}$ can be approximated in the norm operator by operators of the form:*

$$\varphi \in X_p \mapsto \sum_{i\in I} \alpha_i(x)\beta_i(\xi)\int_V \theta_i(\xi_\star)\varphi(x,\xi_\star)\mathrm{d}\nu(\xi_\star) \in X_p \tag{2.6}$$

where I is finite, $\alpha_i \in L^\infty(\Omega)$, $\beta_i \in L^p(V, \mathrm{d}\nu(\xi))$ and $\theta_i \in L^q(V, \mathrm{d}\nu(\xi))$, $1/p + 1/q = 1$.

Remark 2.5 *Since $1 < p < \infty$, one notes that the set $\mathcal{C}_c(V)$ of continuous functions with compact support in V is dense in $L^q(V, \mathrm{d}\nu(\xi))$ as well as in $L^p(V, \mathrm{d}\nu(\xi))$ ($1/p + 1/q = 1$). Consequently, one may assume in the above definition that $\beta_i(\cdot)$ and $\theta_i(\cdot)$ are continuous functions with compact supports in V.*

We end this section with a simple generalization of the classical Riemann–Lebesgue Lemma we shall invoke often in the sequel.

Lemma 2.6 *Let $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ be compactly supported on some interval $]a, b[\subset \mathbb{R}$ $(a < b < \infty)$. Let $\omega(\cdot)$ be a bijective and continuously differentiable function on $\mathbb{R}$ whose derivative admits a finite number of zeros on $]a, b[$. Then,*

$$\lim_{|\xi| \to \infty} \int_{\mathbb{R}} e^{i\xi\omega(x)}\, f(x)\, \mathrm{d}x = 0.$$

`Proof:` Let us denote by $\omega'(\cdot)$ the derivative of $\omega(\cdot)$ and assume, without loss of generality, that there is a unique $x_0 \in \mathbb{R}$ such that $\omega'(x_0) = 0$. Let $\varepsilon > 0$ be fixed. Since $f \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$, there exists $\delta = \varepsilon/2\|f\|_\infty > 0$ such that

$$\sup_{\xi \in \mathbb{R}} \left| \int_{x_0-\delta}^{x_0+\delta} e^{i\xi\omega(x)} f(x) \mathrm{d}x \right| \leqslant \varepsilon.$$

Consequently, it is enough to prove that, for sufficiently large $|\xi|$,

$$\left| \int_{x \leqslant x_0-\delta} e^{i\xi\omega(x)}\, f(x)\, \mathrm{d}x \right| + \left| \int_{x \geqslant x_0+\delta} e^{i\xi\omega(x)}\, f(x)\, \mathrm{d}x \right| \leqslant 2\varepsilon. \tag{2.7}$$

Let us deal with the first integral. Since $\omega(\cdot)$ is bijective and f is compactly supported in $]a, b[$,

$$\int_{x \leqslant x_0-\delta} e^{i\xi\omega(x)}\, f(x)\, \mathrm{d}x = \int_{x_0-\delta}^{b} e^{i\xi\omega(x)}\, f(x)\, \mathrm{d}x = \int_I e^{i\xi y} f(\omega^{-1}(y)) \frac{\mathrm{d}y}{\omega'(\omega^{-1}(y))}$$

where $I = \omega^{-1}([x_0 - \delta, b])$ is a compact interval. Now, since $\omega' \neq 0$ on $[x_0 - \delta, b]$, it is clear that

$$G(\cdot) := f(\omega^{-1}(\cdot)) \frac{1}{\omega'(\omega^{-1}(\cdot))} \in L^1(I)$$

and the (classical) Riemann–Lebesgue Lemma asserts that

$$\lim_{|\xi| \to \infty} \int_I e^{i\xi y} G(y) \mathrm{d}y = 0.$$

One proceeds in the same way with the second integral of (2.7) and this ends the proof. ■

3 On the Rotenberg model

3.1 Statement of the result

We first consider a model of growing cell populations proposed by Rotenberg in 1983 [26] as an improvement of the Lebowitz–Rubinow model [14]. Each cell is characterized by its *degree of maturity* μ and its

maturation velocity $v = \frac{d\mu}{dt}$. The degree of maturation is defined so that $\mu = 0$ at birth (*daughter-cells*) and $\mu = 1$ when the cell divides by mitosis (*mother-cells*). The second variable v is considered as an independent variable within $]a, b[$ $(0 \leqslant a < b \leqslant \infty)$. Denote by $f(t, \mu, v)$ the density of cells having the degree of maturity μ and maturation velocity v at time $t \geqslant 0$. It satisfies the following transport-like equation:

$$\frac{\partial f}{\partial t}(t, \mu, v) + v\frac{\partial f}{\partial \mu}(t, \mu, v) + \sigma(\mu, v) f(t, \mu, v) = \int_a^b r(\mu, v, v') f(t, \mu, v') dv' \qquad (\mu, v) \in]0, 1[\times]a, b[, t \geqslant 0; \quad (3.1)$$

where the kernel $r(\mu, v, v')$ is the transition rate at which cells change their velocities from v' to v and $\sigma(\mu, v)$ denotes the *mortality rate*. During the mitosis, three different situations may occur. First, one can assume that there is a positive correlation $k(v, v') \geqslant 0$ between the maturation velocity v' of a "mother-cell" and the one v of a "daughter-cell". In this case the reproduction rule is given by

$$vf(t, 0, v) = \alpha \int_a^b k(v, v') f(t, 1, v') v' dv' \qquad v \in (a, b), \quad (3.2)$$

where $\alpha \geqslant 0$ is the average number of viable daughters per mitosis. Second, one can assume that daughter cells perfectly inherit their maturation velocity from mother (*perfect memory*), i.e. $v = v'$, or equivalently $k(v, v') = \delta(v - v')$ where $\delta(\cdot)$ denotes the Dirac mass at zero. Then, the biological reproduction rule reads:

$$f(t, 0, v) = \beta(v)\ f(t, 1, v) \qquad v \in]a, b[,$$

where $\beta(v) \geqslant 0$ denotes the average number of viable daughters per mitosis. Finally, one can combine the two previous transition rules which leads to the general reproduction rule we will investigate in the sequel:

$$f(t, 0, v) = \beta(v)\, f(t, 1, v) + \frac{\alpha}{v} \int_a^b k(v, v') f(t, 1, v') v' dv' \qquad v \in]a, b[. \quad (3.3)$$

Of course, one has to complement (3.1) and (3.3) with an initial condition

$$f(0, \mu, v) = f_0(\mu, v) \qquad v \in]a, b[,\ \mu \in]0, 1[\quad (3.4)$$

where $f_0 \in X_p = L^p(]0, 1[\times]a, b[; d\mu dv)$ $(1 < p < \infty)$. The above model has been numerically solved by Rotenberg [26]. The first theoretical approach of this model can be found in the monograph [32, Chapter XIII]. Later, this model has been investigated in [4, 5]. The asymptotic behavior of the solution to (3.1)–(3.4) has been dealt with in [10] for diffuse boundary conditions (3.2) and for a smooth initial data. We will generalize the result of [10] by dealing with the more general reproduction rule (3.3) and by showing the stability of the essential spectrum. Let us make the following assumptions:

(H1) The collision operator

$$B\ : \phi \mapsto B\phi(\mu, v) = \int_a^b r(\mu, v, v') \phi(\mu, v') dv'$$

is a bounded and nonnegative operator in X_p $(1 < p < \infty)$.

(H2) The mortality rate $\sigma(\cdot, \cdot)$ is bounded and nonnegative on $]0, 1[\times]a, b[$. We denote by $\underline{\sigma} = \inf\{\sigma(\mu, v)\ ;\ \mu \in]0, 1[, v \in]a, b[\}$.

(H3) The kernel $k(\cdot,\cdot)$ is nonnegative and such that the mapping

$$K\ :\ f \in Y_p \mapsto \frac{\alpha}{v}\int_a^b k(v,v')f(v')v'dv' \in Y_p$$

is compact, where $Y_p = L^p(]a,b[, v\mathrm{d}v)$ $(1 < p < \infty)$.
(H4) $0 \leqslant \beta(v) \leqslant \beta_0 < 1$ and $\alpha \geqslant 0$.

Let us define the boundary operator $H \in \mathfrak{B}(Y_p)$ by

$$Hf(v) = \beta(v)f(v) + \frac{\alpha}{v}\int_a^b k(v,v')f(v')v'dv' = \beta(v)f(v) + Kf(v) \qquad f \in Y_p. \tag{3.5}$$

Define the unbounded operator $\mathbf{A}_H$ by

$$\mathbf{A}_H\phi(\mu,v) = -v\frac{\partial\phi}{\partial\mu}(\mu,v) - \sigma(\mu,v)\phi(\mu,v)$$

with domain $\mathrm{Dom}(\mathbf{A}_H)$ given by

$$\{\phi \in X_p \text{ such that } \mathbf{A}_H\phi \in X_p, \phi(0,v) \text{ and } \phi(1,v) \in Y_p \text{ and satisfy (3.3)}\}.$$

Note that, since $K \in \mathfrak{C}(Y_p)$ and $\beta_0 < 1$, [19, Theorem 6.8] implies the following:

Theorem 3.1 *Assume* $(\mathrm{H1}) - (\mathrm{H4})$ *to be fulfilled. Then,* $\mathbf{A}_H$ *generates a nonnegative* c_0*–semigroup* $(U(t))_{t\geqslant 0}$ *of* X_p $(1 < p < \infty)$*. As a consequence,* $\mathbf{A}_H + B$ *is also the generator of a* c_0*–semigroup* $(V(t))_{t\geqslant 0}$ *of* X_p*.*

Remark 3.2 *Note that a complete description of the spectrum of* $\mathbf{A}_H$ *is provided in [18].*

Concerning the asymptotic behavior of $(V(t))_{t\geqslant 0}$, one states the following:

Theorem 3.3 *Let* $1 < p < \infty$ *and let* $B \in \mathfrak{B}(X_p)$ *be regular. Then,* $V(t) - U(t)$ *is compact for any* $t \geqslant 0$ *and* $\sigma_{\mathrm{ess}}(U(t)) = \sigma_{\mathrm{ess}}(V(t))$ *for any* $t \geqslant 0$*. In particular,* $\sigma_{\mathrm{ess}}(\mathbf{A}_H + B) = \sigma_{\mathrm{ess}}(\mathbf{A}_H) = \sigma(\mathbf{A}_H)$*.*

Remark 3.4 *We point out that Theorem 3.3 covers all the possible choice of the parameters* a, *and* b*, namely* $0 \leqslant a \leqslant b \leqslant \infty$*.*

3.2 Proof of Theorem 3.3

All this section is devoted to the proof of Theorem 3.3. As a first step, one sees that the mortality rate does not play any role in the compactness of the remainder $R_1(t)$. Indeed, let $\widetilde{\mathbf{A}_H}$ stands for the operator $\mathbf{A}_H$ associated to the constant mortality rate $\underline{\sigma}$. Since $\widetilde{\mathbf{A}_H} - A_H$ is the multiplication operator by the nonnegative function $\sigma(\cdot,\cdot) - \underline{\sigma}$, the Dyson–Phillips formula (2.2) implies that $U_H(t) \leqslant \widetilde{U_H}(t)$ for any $t \geqslant 0$ where $(\widetilde{U_H}(t))_{t\geqslant 0}$ is the c_0–semigroup generated by $\widetilde{\mathbf{A}_H}$. The same occurs for the semigroup $(\widetilde{V_H}(t))_{t\geqslant 0}$ generated by $\widetilde{\mathbf{A}_H} + \mathcal{K}$. Consequently, the first remainder terms $R_1(t)$ and $\widetilde{R_1(t)}$ are such that

$$R_1(t) \leqslant \widetilde{R_1(t)} \qquad \forall\, t \geqslant 0.$$

By a domination argument [9], the compactness of $\widetilde{R_1(t)}$ implies that of $R_1(t)$. Therefore, in order to apply Theorem 2.1, one may assume without loss of generality that

$$\sigma(\mu,v) = -\underline{\sigma} \qquad \forall\,(\mu,v) \in]0,1[\times]a,b[.$$

Now, we point out that it suffices to prove Theorem 3.3 for contractive boundary operator $\|H\| < 1$. Indeed, if $\|H\| \geqslant 1$, recall [19] that the semigroup $(U(t))_{t\geqslant 0}$ enjoys the following similarity property: There exists $q \in (0,1)$ such that

$$U(t) = M_q^{-1} U_q(t) M_q \qquad t \geqslant 0$$

where $M_q \in \mathfrak{B}(X_p)$ is invertible (see [19] for details) and $(U_q(t))_{t\geqslant 0}$ is the c_0–semigroup generated by:

$$\begin{cases} \mathbf{A}_{H_q} \;: \quad \mathrm{Dom}(\mathbf{A}_{H_q}) \subset X_p \to X_p \\ \qquad\qquad \varphi \mapsto \mathbf{A}_{H_q}\varphi(\mu,v) = -v\frac{\partial\varphi}{\partial\mu}(\mu,v) - (\underline{\sigma} + \ln q)\varphi(\mu,v) \end{cases}$$

(note that the collision frequency associated to $\mathbf{A}_{H_q}$ is constant) where the boundary operator H_q is given by $H_q\varphi(v) = H(\exp\{\ln q/v\}\varphi)(v)$. In particular, $q \in (0,1)$ is such that $\|H_q\| < 1$. With obvious notations, one has

$$R_1(t) = M_q^{-1} R_{1,q}(t) M_q$$

and it suffices to prove the compactness of $R_{1,q}(t)$. From now, we will assume that

$$\|H\| < 1.$$

Let us now explicit the resolvent of $\mathbf{A}_H$. To this aim, for any Re$\lambda > -\underline{\sigma}$, define

$$\begin{cases} M_\lambda : \quad Y_p \longrightarrow Y_p \\ \qquad\qquad u \longmapsto M_\lambda u(v) = u(v)\exp\{-\dfrac{\lambda+\underline{\sigma}}{v}\}, \end{cases}$$

$$\begin{cases} \Xi_\lambda : \quad Y_p \longrightarrow X_p \\ \qquad\qquad u \longmapsto \Xi_\lambda u(\mu,v) = u(v)\exp\{-\dfrac{\mu}{v}(\lambda+\underline{\sigma})\}, \end{cases}$$

$$\begin{cases} G_\lambda : \quad X_p \longrightarrow Y_p \\ \qquad\qquad \varphi \longmapsto G_\lambda\varphi(v) = \dfrac{1}{v}\displaystyle\int_0^1 \varphi(\mu',v_\star)\exp\{-\dfrac{1-\mu'}{v}(\lambda+\underline{\sigma})\}\mathrm{d}\mu' \end{cases}$$

and

$$\begin{cases} C_\lambda : \quad X_p \longrightarrow X_p \\ \qquad\qquad \varphi \longmapsto C_\lambda\varphi(\mu,v) = \dfrac{1}{v}\displaystyle\int_0^\mu \varphi(\mu',v)\exp\{-\dfrac{\mu-\mu'}{v}(\lambda+\underline{\sigma})\}\mathrm{d}\mu'. \end{cases}$$

The resolvent of $\mathbf{A}_H$ is given by the following, whose proof can be easily adapted from [19].

Proposition 3.5 *Let $H \in \mathfrak{B}(Y_p)$ be given by* (3.5) *where* (H1) – (H4) *are fulfilled. Then* $\{\lambda \in \mathbb{C}\,;\, Re\lambda > -\underline{\sigma}\} \subset \rho(\mathbf{A}_H)$ *and*

$$(\lambda - \mathbf{A}_H)^{-1} = \Xi_\lambda H (I - M_\lambda H)^{-1} G_\lambda + C_\lambda \qquad Re\lambda > -\underline{\sigma}. \tag{3.6}$$

An important fact to be noticed is that, though Theorem 2.1 is a purely hilbertian result, it turns out to be useful for the treatment of neutron transport problems in L^p–spaces for any $1 < p < \infty$. The reason is the following. Let $1 < p < \infty$ be fixed. We first note that $R_1(t)$ depends continuously on the boundary operator $H \in \mathfrak{B}(Y_p)$. Recalling that $H = \beta\mathrm{Id} + K$ where K is a compact operator on Y_p, it suffices to prove the compactness of $R_1(t)$ for a *finite rank* operator K, i.e. we can assume without loss of generality that the kernel $k(v, v')$ is a degenerate kernel of the form:

$$k(v, v') = \sum_{j \in J} g_j(v) k_j(v') \tag{3.7}$$

where $J \subset \mathbb{N}$ is finite, $g_j(\cdot) \in L^p(]a, b[, v\mathrm{d}v)$ and $k_j(\cdot) \in L^q(]a, b[, v\mathrm{d}v)$ $(1/p + 1/q = 1)$. Moreover, by density, one may assume that $g_j(\cdot)$ and $k_j(\cdot)$ are continuous functions with compact supports on $]a, b[$. In this case, one notes easily that $H \in \mathfrak{B}(Y_r)$ for any $1 < r < \infty$ and the same occurs for $(\lambda - \mathbf{A}_H)^{-1}$ according to Proposition 4.5. The Trotter–Kato Theorem implies then that, for any $t \geqslant 0$, $U_H(t) \in \bigcap_{1<r<\infty} \mathfrak{B}(X_r)$. Similarly, since B is regular and $R_1(t)$ depends continuously on $B \in \mathfrak{B}(X_p)$, one may assume that B is of the form (2.6) where, according to Remark 2.5, the functions β_i and θ_i are continuous with compact supports in $]a, b[$. In this case, it is easy to see that $B \in \bigcap_{1<r<\infty} \mathfrak{B}(X_r)$ so that the same occurs for $R_1(t)$:

$$R_1(t) \in \bigcap_{1<r<\infty} \mathfrak{B}(X_r).$$

Consequently, by an interpolation argument, if $R_1(t)$ is a compact operator on X_2, then $R_1(t)$ is compact on X_p for any $1 < p < \infty$. With this procedure, we may restrict ourselves to prove the compactness of $R_1(t)$ in X_2. In this case, one has the following Proposition whose proof is postponed to the Appendix of this paper.

Proposition 3.6 *Let us assume that $p = 2$. Then, for any regular operator $B \in \mathfrak{B}(X_2)$ and any $\mathrm{Re}\lambda > -\underline{\sigma}$:*

$$\lim_{|\mathrm{Im}\lambda| \to \infty} \left(\|B^\star(\lambda - \mathbf{A}_H)^{-1} B\|_{\mathfrak{B}(X_2)} + \|B(\lambda - \mathbf{A}_H)^{-1} B^\star\|_{\mathfrak{B}(X_2)} \right) = 0.$$

`Proof of Theorem 3.3:` We already saw that it suffices to prove the result for $p = 2$. Proposition 3.6 asserts that Property 2.5 of Theorem 2.1 is fulfilled. Moreover, according to [12, Theorem 3.1], $B(\lambda - \mathbf{A}_H)^{-1}$ is compact for any $\mathrm{Re}\lambda > -\underline{\sigma}$. Since $\|H\| < 1$, $\mathbf{A}_H$ is dissipative (see [3]), Theorem 2.1 asserts that $R_1(t) \in \mathfrak{C}(X_2)$ and the conclusion follows. The identity $\sigma_{\mathrm{ess}}(\mathbf{A}_H + B) = \sigma_{\mathrm{ess}}(\mathbf{A}_H) = \sigma(\mathbf{A}_H)$ follows from Remark 2.3 and [18]. ■

4 On the mono-energetic transport equation in spherical geometry

4.1 Statement of the result

In this section we consider a one-velocity linear transport operator with Maxwell–type boundary conditions in a spherical medium of radius R. For this kind of geometry, neutron transport equation reads [1, Chapter 1]:

$$\frac{\partial \phi}{\partial t}(r, \mu, t) + \mu \frac{\partial \phi}{\partial r}(r, \mu, t) + \frac{1 - \mu^2}{r} \frac{\partial \phi}{\partial \mu}(r, \mu, t) + \Sigma(r, \mu)\phi(r, \mu, t) = \mathcal{K}\phi(r, \mu, t)$$

with the boundary condition

$$\phi(R,\mu,t)=\gamma(-\mu)\phi(R,-\mu,t)+\int_0^1\kappa(\mu,\mu')\phi(R,\mu',t)\mu'd\mu' \qquad -1<\mu<0, \tag{4.1}$$

where r is the distance from the center of the sphere and μ is the cosine of the angle the particle velocity makes with the radius vector, i.e. $(r,\mu)\in[0,R]\times[-1,1]$. The operator $\mathcal{K}$ is a bounded positive operator in $X_p=L^p([0,R]\times[-1,1],r^2drd\mu)\quad(1<p<\infty)$. We make the general assumptions:

i) The collision frequency $\Sigma(\cdot,\cdot)$ is bounded and nonnegative on $[0,R]\times[-1,1]$.

ii) The kernel $\kappa(\cdot,\cdot)$ is nonnegative and such that the mapping

$$\mathbf{K}\ :\ f\mapsto\int_0^1\kappa(\mu,\mu')f(\mu')\mu'd\mu'\ \in\mathfrak{C}(L^p([-1,0],|\mu|d\mu),L^p([0,1],|\mu|d\mu)).$$

iii) The reflective coefficient $\gamma(\cdot)$ is measurable and $0\leqslant\gamma(\mu)\leqslant\gamma_0<1$.

iv) The collision operator $\mathcal{K}\in\mathfrak{B}(X_p)$ is regular $(1<p<\infty)$.

Define the boundary operator $\mathbf{H}=\mathbf{J}+\mathbf{K}$ where

$$\mathbf{J}f(\mu)=\gamma(\mu)f(-\mu)\qquad\forall\mu\in(0,1),\ f\in L^p([-1,0],|\mu|d\mu)$$

and the transport operator $\mathbf{A_H}$ by

$$\mathbf{A_H}\phi(r,\mu)=-\mu\frac{\partial\phi}{\partial r}(r,\mu)-\frac{1-\mu^2}{r}\frac{\partial\phi}{\partial\mu}(r,\mu)-\Sigma(r,\mu)\phi(r,\mu)$$

with domain $\mathrm{Dom}(\mathbf{A_H})$ equals to

$$\{\phi\in X_p \text{ such that } \mathbf{A_H}\phi\in X_p,\phi(R,\mu)\in L^p([-1,0],|\mu|d\mu) \text{ and satisfies (4.1)}\}.$$

The main properties of the transport operator $\mathbf{A_H}$ for various boundary operator $\mathbf{H}$ has been dealt with in [34, 35] and it spectrum has been described in full generality in [18] for Maxwell–like boundary operator $\mathbf{H}$ satisfying assumptions ii)–iii). In particular, a consequence of [19, Theorem 6.8] is the following generation result:

Theorem 4.1 *Assume $i)-iv)$ to be fulfilled. Then, $\mathbf{A_H}$ generates a nonnegative c_0–semigroup $(U(t))_{t\geqslant0}$ of X_p $(1<p<\infty)$. As a consequence, $\mathbf{A_H}+\mathcal{K}$ is also the generator of a c_0–semigroup $(V(t))_{t\geqslant0}$ of X_p.*

Remark 4.2 *Let us say a few words about the proof of Theorem 4.1. As it is well–known [18] (see also Section below), up to a suitable change of variables, $\mathbf{A_H}$ is similar to a one–velocity transport operator T_H acting on some Banach space $\mathcal{X}_p$ (see* (4.5) *below for details). Under assumptions $i)-iv)$, it is then a direct consequence of [19, Theorem 6.8] that T_H generates a c_0–semigroup in $\mathcal{X}_p$ $(1<p<\infty)$. This implies obviously that $\mathbf{A_H}$ is a generator of a c_0–semigroup in X_p.*

The main result of this section is then the following:

Theorem 4.3 *Let $1<p<\infty$ and let $\mathcal{K}\in\mathfrak{B}(X_p)$ be regular. Then $V(t)-U(t)$ is compact for any $t\geqslant0$ and $\sigma_{\mathrm{ess}}(U(t))=\sigma_{\mathrm{ess}}(V(t))\ \forall t\geqslant0$. Moreover, $\sigma_{\mathrm{ess}}(\boldsymbol{A}_{\mathbf{H}}+\mathcal{K})=\sigma_{\mathrm{ess}}(\boldsymbol{A}_{\mathbf{H}})=\sigma(\boldsymbol{A}_{\mathbf{H}})$.*

Remark 4.4 *A very precise description of $\sigma(\boldsymbol{A}_{\mathbf{H}})$ can be found in [18].*

4.2 Proof of Theorem 4.3

The method of the proof is very similar to that used in the proof of Theorem 3.3 and consists in applying Theorem 2.1. We resume briefly some of the arguments developed in Section 3.2. Define $R_1(t) = V(t) - U(t)$ for any $t \geqslant 0$. The proof consists in proving that $R_1(t) \in \mathfrak{C}(X_p)$ for any $t \geqslant 0$. Since $\mathcal{K}$ is a nonnegative operator, it is easy to see that it suffices to prove the result for a constant collision frequency, say

$$\Sigma(r,\mu) = \Sigma \qquad \text{for any } (r,\mu) \in [0,R] \times [-1,1]. \tag{4.2}$$

Moreover, one may assume without loss of generality that

$$\|\mathbf{H}\| < 1.$$

As above, since $\mathcal{K}$ is regular and $\mathbf{K}$ is compact, it suffices to prove the result for a collision operator of the form

$$\mathcal{K}\varphi(r,\mu) = \sum_{i \in I} \alpha_i(r)\beta_i(\mu) \int_{-1}^{1} \theta_i(\mu_\star)\varphi(r,\mu_\star)\mathrm{d}\mu_\star \tag{4.3}$$

where $I \subset \mathbb{N}$ is finite, $\alpha_i(\cdot) \in L^\infty([0,R])$ and $\beta_i(\cdot),\ \theta_i(\cdot) \in \mathcal{C}_c(]-1,1[)$ $(i \in I)$ and for a kernel $\kappa(\cdot,\cdot)$ which reads

$$\kappa(\mu,\mu') = \sum_{j \in J} \mathbf{g}_j(\mu)\mathbf{k}_j(\mu') \tag{4.4}$$

where $J \subset \mathbb{N}$ is finite, $\mathbf{g}_j(\cdot) \in \mathcal{C}_c(]0,1[)$ and $\mathbf{k}_j \in \mathcal{C}_c(]-1,0[)$, $j \in J$. In such a case, $R_1(t) \in \bigcap_{1<r<\infty} \mathfrak{B}(X_p)$ so that it suffices to prove the compactness of $R_1(t)$ for $p = 2$. Throughout the sequel, we will therefore restrict ourselves to the case $p = 2$ and will assume (4.2), (4.3) and (4.4) to be satisfied. At this point it is convenient to use a change of variable already performed in [18] (see also [27, 34, 35]): let $x = r\mu$ and $y = r\sqrt{1-\mu^2}$. This transformation is one–to–one from $[0,R] \times [-1,1]$ onto $\Omega = \{(x,y);\ x^2 + y^2 \leqslant R, 0 \leqslant y \leqslant R\}$. Then, there exists an isometric isomorphism $\mathcal{J}$ from X_2 to $\mathcal{X}_2 := L^2(\Omega, ydydx)$ defined as

$$\begin{cases} \mathcal{J}\ :\quad L^2([0,R]\times[-1,1], r^2drd\mu) \to \mathcal{X}_2 \\ \qquad\quad \phi(r,\mu) \mapsto \mathcal{J}\phi(x,y) = \phi(\sqrt{x^2+y^2}, x/\sqrt{x^2+y^2}), \qquad (x,y) \in \Omega. \end{cases}$$

In this case, the transport operator $\mathbf{A_H} = \mathcal{J}^{-1}T_H\mathcal{J}$ where T_H is the following transport operator:

$$\begin{cases} T_H\ :\quad \mathrm{Dom}(T_H) \to \mathcal{X}_2 \\ \qquad\quad \varphi \mapsto T_H\varphi(x,y) = -\dfrac{\partial \varphi}{\partial x}(x,y) - \Sigma\varphi(x,y), \end{cases} \tag{4.5}$$

whose domain $\mathrm{Dom}(T_H)$ is

$$\{\psi \in \mathcal{X}_2 \text{ such that } T_H\varphi \in \mathcal{X}_2; \psi(y_\pm, y) \in Y_2 \text{ and } \psi(y_-,y) = H\,\psi(y_+,y)\}$$

where $y_\pm = \pm\sqrt{R^2 - y^2}$ $(y \in S = [0,R])$ and $Y_2 = L^2(S, y\mathrm{d}y)$. The boundary operator $H \in \mathfrak{B}(Y_2)$ is given by $H = J + K$ where $J\varphi(y) = \alpha(y)\varphi(y)$ and

$$K\varphi(-\sqrt{R^2-y^2}, y) = \int_0^R k(y,y')\varphi(-\sqrt{R^2-y'^2}, y')\frac{y'}{R}dy' \qquad (y \in S)$$

where the new scattering kernel $k(\cdot,\cdot)$ and the reflective coefficient $\alpha(\cdot)$ are defined as

$$k(y,y') = \kappa(-\sqrt{R^2-y^2}/R, \sqrt{R^2-y^2}/R), \qquad \alpha(y) = \gamma(-\sqrt{R^2-y^2}/R).$$

Note that, since $\|\mathbf{H}\| < 1$ and $\mathcal{J}$ is isometric, one has $\|H\| < 1$ so that T_H is dissipative.

In the same way, one can define the following collision operator $\mathcal{B} = \mathcal{J}\mathcal{K}\mathcal{J}^{-1} \in \mathfrak{B}(\mathcal{X}_2)$. Straightforward computations yield

$$\mathcal{B}\varphi(x,y) = \sum_{i\in I} \alpha_i(\sqrt{x^2+y^2})\beta_i(x/\sqrt{x^2+y^2}) \times \\ \times \int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} \theta_i(z/\sqrt{x^2+y^2})\varphi(z,\sqrt{x^2+y^2-z^2})\frac{\mathrm{d}z}{\sqrt{x^2+y^2}}. \quad (4.6)$$

Let us denote by $(\mathcal{U}(t))_{t\geqslant 0}$ and $(\mathcal{V}(t))_{t\geqslant 0}$ the c_0–semigroups in $\mathcal{X}_2$ generated by T_H and $T_H + \mathcal{B}$ respectively. Since $R_1(t) = \mathcal{J}^{-1}(\mathcal{V}(t) - \mathcal{U}(t))\mathcal{J}$ for any $t \geqslant 0$, one has to prove that $\mathcal{V}(t) - \mathcal{U}(t) \in \mathfrak{C}(\mathcal{X}_2)$. To this aim we shall apply Theorem 2.1 and we have to compute explicitly the resolvent of T_H. Define for any $\mathrm{Re}\lambda > -\Sigma$:

$$\begin{cases} M_\lambda : \quad Y_2 \longrightarrow Y_2 \\ \qquad u \longmapsto M_\lambda u(y) = u(y)\exp\{-2(\lambda+\Sigma)\sqrt{R^2-y^2}\}, \quad (y \in S) \end{cases}$$

$$\begin{cases} \Xi_\lambda : \quad Y_2 \longrightarrow \mathcal{X}_2 \\ \qquad u \longmapsto \Xi_\lambda u(x,y) = u(y)\exp\{-(\lambda+\Sigma)(x+\sqrt{R^2-y^2})\}, \end{cases}$$

$$\begin{cases} G_\lambda : \quad \mathcal{X}_2 \longrightarrow Y_2 \\ \qquad \varphi \longmapsto G_\lambda\varphi(y) = \displaystyle\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} \varphi(z,y)e^{-(\lambda+\Sigma)(\sqrt{R^2-y^2}-z)}\mathrm{d}z \end{cases}$$

and

$$\begin{cases} C_\lambda : \quad \mathcal{X}_2 \longrightarrow \mathcal{X}_2 \\ \qquad \varphi \longmapsto C_\lambda\varphi(x,y) = \displaystyle\int_{-\sqrt{R^2-y^2}}^{x} \varphi(z,y)e^{-(\lambda+\Sigma)(x-z)}\mathrm{d}z. \end{cases}$$

The resolvent of T_H is then given by the following (see [18])

Proposition 4.5 *Let $H \in \mathfrak{B}(Y_2)$ be given as above. Then*

$$(\lambda - T_H)^{-1} = \Xi_\lambda H(I - M_\lambda H)^{-1}G_\lambda + C_\lambda \qquad \forall\, Re\lambda > -\Sigma.$$

Then, the key point of the proof of Theorem 4.3 stands in the following whose proof is given in the Appendix 7:

Proposition 4.6 *For any regular operator $B \in \mathfrak{B}(\mathcal{X}_2)$ and any $\mathrm{Re}\lambda > -\Sigma$:*

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \left(\|B^\star(\lambda - T_H)^{-1}B\|_{\mathfrak{B}(\mathcal{X}_2)} + \|B(\lambda - T_H)^{-1}B^\star\|_{\mathfrak{B}(\mathcal{X}_2)}\right) = 0.$$

`Proof of Theorem 4.3:` The proof of Theorem 4.3 is now a straightforward application of Theorem 2.1 as in Theorem 3.3. ■

5 Transport equations in slab geometry

Let us consider the following transport equation in a slab with thickness $2a > 0$:

$$\frac{\partial \varphi}{\partial t}(x,\xi,t) + \xi \frac{\partial \varphi}{\partial x}(x,\xi,t) + \sigma(x,\xi)\varphi(x,\xi,t) = \int_{-1}^{1} \kappa(x,\xi,\xi_\star)\varphi(x,\xi_\star,t)\mathrm{d}\xi_\star \tag{5.1}$$

with the boundary conditions

$$\varphi^i = H(\varphi^o) \tag{5.2}$$

and the initial datum $\varphi(x,\xi,t=0) = \phi_0(x,\xi) \in X_p = L^p([-a,a]\times[-1,1]; \mathrm{d}x\mathrm{d}\xi)$ $(1 < p < \infty)$. The incoming boundary of the phase space D^i and the outgoing one D^o are given by :

$$D^i := D_1^i \cup D_2^i := \{-a\}\times[0,1] \cup \{a\}\times[-1,0],$$

$$D^o := D_1^o \cup D_2^o := \{-a\}\times[-1,0] \cup \{a\}\times[0,1],$$

while the associated boundary spaces are

$$X_p^i := L^p(D_1^i, |\xi|d\xi) \times L^p(D_2^i, |\xi|d\xi) = X_{1,\,p}^i \times X_{2,\,p}^i,$$

and

$$X_p^o := L^p(D_1^o, |\xi|d\xi) \times L^p(D_2^o, |\xi|d\xi) = X_{1,\,p}^o \times X_{2,\,p}^o,$$

endowed with their natural norms (see [13] for details). Let W_p be the partial Sobolev space $W_p := \{\psi \in X_p$ such that $\xi\frac{\partial \psi}{\partial x} \in X_p\}$. Any function $\psi \in W_p$ admits traces on D^o and D^i denoted by ψ^o and ψ^i respectively. Precisely, $\psi^o = (\psi_1^o, \psi_2^o)$ and $\psi^i = (\psi_1^i, \psi_2^i)$ are given by

$$\begin{cases} \psi_1^o(\xi) = \psi(-a,\xi) & \xi \in (-1,0); \\ \psi_2^o(\xi) = \psi(a,\xi) & \xi \in (0,1); \\ \psi_1^i(\xi) = \psi(-a,\xi) & \xi \in (0,1); \\ \psi_2^i(\xi) = \psi(a,\xi) & \xi \in (-1,0). \end{cases} \tag{5.3}$$

We describe the boundary operator H relating the incoming flux ψ^i to the outgoing one ψ^o by

$$\begin{cases} H \ : X_{1,\,p}^o \times X_{2,\,p}^o \ \to X_{1,\,p}^i \times X_{2,\,p}^i \\ H\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{cases}$$

where $H_{jk} \in \mathcal{L}(X_{k,p}^o; X_{j,p}^i); j,k = 1,2$. Let us now define the transport operator associated to the boundary conditions induced by H

$$\begin{cases} T_H \ : \mathrm{Dom}(T_H) \subset X_p \to X_p \\ \qquad \psi \mapsto T_H\psi(x,\xi) = -\xi\dfrac{\partial \psi}{\partial x}(x,\xi) - \sigma(x,\xi)\psi(x,\xi), \end{cases}$$

where

$$\mathrm{Dom}(T_H) = \{\psi \in W_p \text{ such that } \psi^o \in X_p^o \text{ and } H\psi^o = \psi^i\}.$$

We make the following assumptions

(H1) The collision frequency $\sigma(\cdot,\cdot)$ is measurable, bounded and nonnegative on $[-a,a]\times[-1,1]$.

(H2) The collision kernel $\kappa(\cdot,\cdot,\cdot)$ is measurable and nonnegative on $[-a,a]\times[-1,1]\times[-1,1]$ and such that the operator

$$\mathcal{K}\ :\ f(x,\xi)\mapsto \mathcal{K}f(x,\xi)=\int_{-1}^{1}\kappa(x,\xi,\xi_\star)f(x,\xi_\star)\mathrm{d}\xi_\star$$

is *regular* on X_p $(1<p<\infty)$.

Concerning the boundary operator H we assume that one of the following assumptions is fulfilled:

a) H is a diagonal operator of the form $H=\begin{pmatrix} H_{11} & 0\\ 0 & H_{22}\end{pmatrix}$ with $H_{11}=\rho_1 J_1+K_1$ and $H_{22}=\rho_2 J_2+K_2$ where ρ_i is positive $(i=1,2)$ and K_i is a compact operator. The operators J_i $(i=1,2)$ are given by

$$\begin{cases} J_1\ :\ & X_{1,p}^0\to X_{1,p}^i\\ & \psi(-a,\cdot)\mapsto J_1\psi(\xi)=\psi(-a,-\xi)\end{cases}$$

$$\begin{cases} J_2\ :\ & X_{2,p}^0\to X_{2,p}^i\\ & \psi(a,\cdot)\mapsto J_2\psi(\xi)=\psi(a,-\xi)\end{cases}$$

b) H is a off–diagonal operator of the form $H=\begin{pmatrix} 0 & H_{12}\\ H_{21} & 0\end{pmatrix}$ with $H_{12}=\beta_1 I_{12}+K_1$ and $H_{21}=\beta_2 I_{21}+K_2$ where β_i is positive $i=1,2$ and K_i is a compact operator. The operators I_{12} and I_{21} are given by

$$\begin{cases} I_{12}\ :\ & X_{2,p}^0\to X_{1,p}^i\\ & \psi(a,\cdot)\mapsto I_{12}\psi(\xi)=\psi(-a,\xi)\end{cases}$$

$$\begin{cases} I_{21}\ :\ & X_{1,p}^0\to X_{2,p}^i\\ & \psi(-a,\cdot)\mapsto I_{21}\psi(\xi)=\psi(a,\xi)\end{cases}$$

c) The boundary operator H is compact.

There is a vast literature dealing with model (5.1)–(5.2) starting with the pioneering work of Lehner and Wing [15]. We only mention the recent results of [13] dealing with the asymptotic behavior of the solution to (5.1)–(5.2) as well as [6, 7] which take into account possibly unbounded collision operator $\mathcal{K}$. In the L^p–setting, our main result generalizes the existing ones:

Theorem 5.1 *Let $1<p<\infty$. Let* (H1) *and* (H2) *be fulfilled. Moreover, assume that H satisfies one of the assumptions a), b) or c). Then, $V(t)-U(t)\in\mathfrak{C}(X_p)$ for any $t\geqslant 0$. In particular, $\sigma_{\mathrm{ess}}(V(t))=\sigma_{\mathrm{ess}}(U(t))$ $(t\geqslant 0)$ where $(V(t))_{t\geqslant 0}$ is the c_0–semigroup generated by $T_H+\mathcal{K}$ and $(U(t))_{t\geqslant 0}$ is the one generated by T_H.*

`Proof:` Note that the existence of the semigroup $(U(t))_{t\geqslant 0}$ generated by T_H is a direct consequence of [19]. To prove that $V(t)-U(t)\in\mathfrak{C}(X_p)$ one sees easily, arguing as above that it suffices to prove the result for $p=2$, $\|H\|<1$ and a constant collision frequency $\sigma(x,\xi)=\sigma$. In this case, one deduces from [11, Theorem 2.1, p. 55], [13, Proposition 3.1], and [11, Theorem 3.2, p. 77] that property (2.5) of Theorem 2.1

holds. The previous references correspond respectively to the assumption a), b) and c). Since $\mathcal{K}(\lambda - T_H)^{-1}$ is compact for $\mathrm{Re}\lambda > -\sigma$ [12], one concludes thanks to Theorem 2.1. ■

Remark 5.2 *Note that, in [8] (see also [6, 7]) the identity $r_{\mathrm{ess}}(V(t)) = r_{\mathrm{ess}}(U(t))$ is established exploiting the explicit nature of $(U(t))_{t\geqslant 0}$ in the case of perfect reflecting boundary conditions or periodic conditions. Even if, for general boundary operator H satisfying a)–c) the semigroup $(U(t))_{t\geqslant 0}$ can also be made explicit (see for instance [17]), the resolvent approach is much more easy to apply and leads to similar results.*

6 Appendix 1: Proof of Proposition 3.6

The aim of this Appendix is to prove the Proposition 3.6. We decompose its proof into several steps. The strategy is inspired by similar results in [13]. First, since B is of the form (2.4), it is enough to show by linearity that

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|B_1(\lambda - \mathbf{A}_H)^{-1}B_2\| = 0 \qquad \forall\, \mathrm{Re}\lambda > -\underline{\sigma}$$

where

$$B_i\varphi(\mu, v) = \alpha_i(\mu)\beta_i(v)\int_a^b \theta_i(v_\star)\varphi(\mu, v_\star)\mathrm{d}v_\star, \qquad (i = 1, 2)$$

and $\alpha_i(\cdot) \in L^\infty(]0,1[,\ \beta_i(\cdot), \theta_i(\cdot) \in \mathcal{C}_c(]a,b[)\ i = 1,2$. This shall be done in several steps. Recall that, by Proposition 4.5,

$$(\lambda - \mathbf{A}_H)^{-1} = \Xi_\lambda H(I - M_\lambda H)^{-1}G_\lambda + C_\lambda \qquad Re\lambda > -\underline{\sigma}.$$

Step 1: We first note that, for any $\mathrm{Re}\lambda > -\underline{\sigma}$ the operator C_λ is nothing else but the resolvent of the transport operator $\mathbf{A}_H$ in the case of absorbing boundary conditions, $H = 0$. Then, according to a result by M. Mokhtar–Kharroubi [21, Lemma 2.1],

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|B_1 C_\lambda B_1\| = 0 \qquad \forall\, \mathrm{Re}\lambda > -\underline{\sigma}.$$

Therefore, it is enough to prove that

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|B_1\Xi_\lambda H(I - M_\lambda H)^{-1}G_\lambda B_2\| = 0 \qquad \forall\, \mathrm{Re}\lambda > -\underline{\sigma}. \tag{6.1}$$

Step 2: We note that, adapting the result of [11, Theorem 3.2, p. 77] (see [10] for details), one has

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|B_1\Xi_\lambda K(I - M_\lambda H)^{-1}G_\lambda B_2\| = 0 \qquad \forall\, \mathrm{Re}\lambda = -\underline{\sigma} + \omega, \omega > 0. \tag{6.2}$$

Step 3. Using the fact that $H = \beta\mathrm{Id} + K$, it remains only to show that

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|B_1\Xi_\lambda(I - M_\lambda H)^{-1}G_\lambda B_2\| = 0 \qquad \forall\, \mathrm{Re}\lambda = -\underline{\sigma} + \omega, \omega > 0.$$

To do, using the fact that $(I - M_\lambda H)^{-1} = \sum_{n=0}^\infty (M_\lambda H)^n$, together with the dominated convergence theorem, it suffices to show that, for any integer $n \in \mathbb{N}$

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|B_1\Xi_\lambda(M_\lambda H)^n G_\lambda B_2\| = 0 \qquad \forall\, \mathrm{Re}\lambda = -\underline{\sigma} + \omega, \omega > 0. \tag{6.3}$$

Since $M_\lambda H = \beta M_\lambda + M_\lambda K$, for any $n \in \mathbb{N}$, $(M_\lambda H)^n = \sum_{j=1}^{2^n} P_j(\lambda)$ where $P_j(\lambda)$ is the product of n factors formed with βM_λ and $M_\lambda K$. Among these factors, only $P_{2^n}(\lambda) = (\beta M_\lambda)^n$ does not involve K whereas, for $j \in \{1, \ldots, 2^n - 1\}$, the operator K appears at least once in the expression of $P_j(\lambda)$.

Step 3.1 : One proves that, for any $j \in \{1, \ldots, 2^n - 1\}$,

$$\lim_{|\mathrm{Im}\lambda| \to \infty} \|P_j(\lambda) G_\lambda B_2\|_{\mathfrak{B}(X_2, Y_2)} = 0 \qquad \forall \mathrm{Re}\lambda = -\underline{\sigma} + \omega, \omega > 0.$$

By assumption, there exists $k \in \{0, \ldots, n-1\}$ such that $P_j(\lambda) = P_j^1(\lambda) M_\lambda K (\beta M_\lambda)^k$ where $P_j^1(\lambda)$ is a product of operators $M_\lambda K$ and βM_λ. As a by–product,

$$\sup\{\|P_j^1(\lambda) M_\lambda\|\,;\, \mathrm{Re}\lambda = -\underline{\sigma} + \omega\} < \infty.$$

It suffices then to show that, for any $k \geqslant 0$,

$$\lim_{|\mathrm{Im}\lambda| \to \infty} \|K(\beta M_\lambda)^k G_\lambda B_2\|_{\mathfrak{B}(X_2, Y_2)} = 0 \qquad \forall \mathrm{Re}\lambda > -\underline{\sigma} + \omega. \tag{6.4}$$

A direct computation shows that

$$K M_\lambda^k G_\lambda B_2 \varphi(v) = \frac{\alpha}{v} \int_a^b k(v, v_\star) \beta_2(v_\star) \exp\{-k(\lambda + \underline{\sigma})/v_\star\} \mathrm{d}v_\star \times$$
$$\times \int_0^1 \exp\{-\frac{(1-\mu')}{v_\star}(\lambda + \underline{\sigma})\} \alpha_2(\mu') \mathrm{d}\mu' \int_a^b \theta_2(w) \varphi(\mu', w) \mathrm{d}w.$$

Then, one may decompose $K M_\lambda^k G_\lambda B_2$ as $K M_\lambda^k G_\lambda B_2 = \mathcal{R}_1(\lambda) \mathcal{R}_2$ with

$$\mathcal{R}_2 \;: \varphi \in X_2 \mapsto \mathcal{R}_2 \varphi(\mu) = \alpha_2(\mu) \int_a^b \theta_2(w) \varphi(\mu, w) \mathrm{d}w \in L^2(]0,1[, \mathrm{d}\mu)$$

and

$$\mathcal{R}_1(\lambda) \psi(v) = \frac{\alpha}{v} \int_a^b k(v, v_\star) \beta_2(v_\star) \exp\{-k(\lambda + \underline{\sigma})/v_\star\} \mathrm{d}v_\star \times$$
$$\times \int_0^1 \exp\{-\frac{(1-\mu')}{v_\star}(\lambda + \underline{\sigma})\} \psi(\mu') \mathrm{d}\mu' \in X_2, \qquad \psi \in L^2(]0,1[, \mathrm{d}\mu).$$

It is then enough to show that $\lim_{|\mathrm{Im}\lambda| \to \infty} \|\mathcal{R}_1(\lambda)\| = 0$ for any $\mathrm{Re}\lambda = -\underline{\sigma} + \omega$. By linearity, using that the kernel $k(v, v_\star)$ is of the form (3.7), one may assume without loss of generality that $k(v, v_\star) = g(v) k(v_\star)$ where both $g(\cdot)$ and $k(\cdot)$ are continuous functions with compact supports in $]a, b[$. Now, let us fix $\psi \in L^2(]0,1[, \mathrm{d}\mu)$ and denote by $\widetilde{\psi}$ its trivial extension to $\mathbb{R}$. Then, one sees easily that

$$\mathcal{R}_1(\lambda) \psi(v) = \frac{g(v)}{v} \int_{\mathbb{R}} F_\lambda(k + 1 - \mu') \widetilde{\psi}(\mu') \mathrm{d}\mu'$$

where

$$F_\lambda(x) = \alpha \int_a^b k(v_\star) \beta_2(v_\star) \exp\{-\frac{x}{v_\star}(\lambda + \underline{\sigma})\} \mathrm{d}v_\star \qquad \forall x \geqslant 0.$$

One sees that

$$\int_0^\infty \sup_{\mathrm{Re}\lambda=-\underline{\sigma}+\omega} |F_\lambda(x)|^2 \mathrm{d}x \leqslant \frac{\alpha^2}{2\omega}\int_a^b |k(v_\star)|^2 v_\star \,\mathrm{d}v_\star \int_a^b |\beta_2(v_\star)|^2 \mathrm{d}v_\star < \infty.$$

According to Riemann–Lebesgue Lemma 2.6 and the Dominated Convergence Theorem, it is not difficult to see that

$$\lim_{|\mathrm{Im}\lambda|\to\infty}\int_0^\infty |F_\lambda(x)|^2\mathrm{d}x = 0 \qquad \forall \mathrm{Re}\lambda = -\underline{\sigma}+\omega.$$

Since

$$\|\mathcal{R}_1(\lambda)\|^2 \leqslant \left(\int_a^b \left|\frac{g(v)}{v}\right|^2 v\mathrm{d}v\right)\int_0^\infty |F_\lambda(x)|^2\mathrm{d}x.$$

this proves the desired result. It remains to investigate the case $j = 2^n$:

Step 3.2 : It remains to evaluate the behavior of $\|B_1\Xi_\lambda P_{2^n} G_\lambda B_2\|$ as $|\mathrm{Im}\lambda|$ goes to infinity, where $P_{2^n} = (\beta M_\lambda)^n$. Precisely, let us show that

$$\lim_{|\mathrm{Im}\lambda|\to\infty}\|B_1\Xi_\lambda(\beta M_\lambda)^n G_\lambda B_2\| = 0 \qquad \forall\, \mathrm{Re}\lambda = -\underline{\sigma}+\omega, \omega>0. \tag{6.5}$$

Let $\varphi \in X_2$. Straightforward calculations yield

$$B_1\Xi_\lambda M_\lambda^n G_\lambda B_2\varphi(\mu,v) = \alpha_1(\mu)\beta_1(v)\int_a^b \frac{\theta_1(v_\star)\beta_2(v_\star)}{v_\star}\exp\{-\frac{(\lambda+\underline{\sigma})}{v_\star}\mu\}\mathrm{d}v_\star \times$$
$$\times \int_0^1 \alpha_2(\mu')\exp\{-\frac{(\lambda+\underline{\sigma})}{v_\star}(n+1-\mu')\}\mathrm{d}\mu' \int_a^b \theta_2(w)\varphi(\mu',w)\mathrm{d}w.$$

As above, one may split this operator as $B_1\Xi_\lambda M_\lambda^n G_\lambda B_2 = \mathcal{A}_3\mathcal{A}_2(\lambda)\mathcal{A}_1$ where

$$\mathcal{A}_1 : \varphi \in X_2 \mapsto \mathcal{A}_1\varphi(\mu) = \alpha_2(\mu)\int_a^b \theta_2(w)\varphi(\mu,w)\mathrm{d}w \in L^2(]0,1[,\mathrm{d}\mu),$$

$$\begin{cases}\mathcal{A}_2(\lambda)\ :\ L^2(]0,1[,\mathrm{d}\mu) \to L^2(]0,1[,\mathrm{d}\mu)\\ \psi \mapsto \mathcal{A}_2(\lambda)\psi(\mu) = \displaystyle\int_a^b \mathrm{d}v_\star \int_0^1 \frac{\theta_1(v_\star)\beta_2(v_\star)}{v_\star}\psi(\mu')e^{-\frac{(\lambda+\underline{\sigma})}{v_\star}(n+1+\mu-\mu')}\mathrm{d}\mu'\end{cases}$$

and

$$\mathcal{A}_3\ :\ \psi \in L^2(]0,1[,\mathrm{d}\mu) \mapsto \alpha_1(\mu)\beta_1(v)\psi(\mu) \in X_2.$$

It is clearly sufficient to prove that

$$\lim_{|\mathrm{Im}\lambda|\to\infty}\|\mathcal{A}_2(\lambda)\| = 0 \qquad \forall\ \mathrm{Re}\lambda = -\underline{\sigma}+\omega.$$

As in the proof of Step 3.1, let us define, for any $x \in \mathbb{R}$

$$F_\lambda(x) = \int_a^b \frac{\theta_1(v_\star)\beta_2(v_\star)}{v_\star}\exp\{-\frac{(\lambda+\underline{\sigma})}{v_\star}x\}\mathrm{d}v_\star$$

so that

$$\mathcal{A}_2(\lambda)\psi(\mu) = \int_{\mathbb{R}} F_\lambda(n+1+\mu-x)\widetilde{\psi}(x)\mathrm{d}x, \qquad \psi \in L^2(]0,1[,\mathrm{d}\mu)$$

where $\widetilde{\psi}$ is the trivial extension to $\mathbb{R}$ of $\psi \in L^2(]0,1[,\mathrm{d}\mu)$. As in the proof of Step 3.1, one can show that

$$\int_{\mathbb{R}} \left(\sup_{\mathrm{Re}\lambda = -\underline{\sigma}+\omega} |F_\lambda(x)|^2 \right) \mathrm{d}x < \infty$$

and

$$\|\mathcal{A}_2(\lambda)\| \leqslant \|F_\lambda(\cdot)\|_{L^2(\mathbb{R})} \qquad (\mathrm{Re}\lambda = -\underline{\sigma}+\omega).$$

Then, applying again Riemmann–Lebesgue Lemma 2.6 together with the dominated convergence theorem, one gets

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|F_\lambda(\cdot)\|_{L^2(\mathbb{R})} = 0 \qquad (\mathrm{Re}\lambda = -\underline{\sigma}+\omega),$$

which leads to the conclusion. Combining all the above steps, we proved Proposition 3.6.

7 Appendix 2: Proof of Proposition 4.6

In this appendix, we prove Proposition 4.6 which is the key point of the proof of Theorem 4.3. Since $\mathcal{B}$ is of the form (4.6), by a linearity argument it suffices to prove that, for any $\omega > 0$,

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|\mathcal{B}_1(\lambda - T_H)^{-1}\mathcal{B}_2\|_{\mathfrak{B}(\mathcal{X}_2)} = 0 \qquad \forall\, \mathrm{Re}\lambda = -\Sigma + \omega \tag{7.1}$$

where

$$\mathcal{B}_i\varphi(x,y) = \frac{\alpha_i(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}\beta_i(x/\sqrt{x^2+y^2})\times \\ \times \int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}} \theta_i(z/\sqrt{x^2+y^2})\varphi(z,\sqrt{x^2+y^2-z^2})\mathrm{d}z$$

where $\alpha_i(\cdot) \in L^\infty([0,R])$, $\beta_i(\cdot) \in \mathcal{C}_c(]-1,0[)$ and $\theta_i(\cdot) \in \mathcal{C}_c(]0,1[)$ $(i=1,2)$. We shall prove (7.1) in several steps.

Step 1: As in the first step of the proof of Proposition 3.6, it is a direct consequence of [21] that

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|\mathcal{B}_1 C_\lambda \mathcal{B}_2\| = 0 \qquad \forall \mathrm{Re}\lambda = -\Sigma + \omega.$$

Step 2: We show now that

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|\mathcal{B}_1 \Xi_\lambda K(I - M_\lambda H)^{-1} G_\lambda \mathcal{B}_2\| = 0 \qquad \forall\, \mathrm{Re}\lambda = -\Sigma + \omega, \omega > 0.$$

Let us first prove that, for any $\varphi \in Y_2$,

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|\mathcal{B}_1 \Xi_\lambda \varphi\|_{\mathcal{X}_2} = 0 \qquad \mathrm{Re}\lambda > -\Sigma. \tag{7.2}$$

One has, for a. e. $(x,y)\in\Omega$:

$$\mathcal{B}_1\Xi_\lambda\varphi(x,y)=\alpha_1(\sqrt{x^2+y^2})\beta_1(x/\sqrt{x^2+y^2})\int_{-\sqrt{x^2+y^2}}^{\sqrt{x^2+y^2}}\theta_1(z/\sqrt{x^2+y^2})$$
$$\varphi(\sqrt{x^2+y^2-z^2})\exp\{-(\lambda+\Sigma)(z+\sqrt{R^2+z^2-x^2-y^2})\}\frac{\mathrm{d}z}{\sqrt{x^2+y^2}},$$

and the Riemann–Lebesgue Lemma implies that, for any $\mathrm{Re}\lambda=-\Sigma+\omega$,

$$\lim_{|\mathrm{Im}\lambda|\to\infty}|\mathcal{B}_1\Xi_\lambda\varphi(x,y)|^2=0\qquad\text{a.e. }(x,y)\in\Omega.$$

Then, the dominated convergence theorem leads to (7.2). Now, let B be the unit ball of $\mathcal{X}_2$. It is clear that,

$$M:=\sup\{\|(I-M_\lambda H)^{-1}G_\lambda\mathcal{B}_2\psi\|\,;\,\psi\in B,\,\mathrm{Re}\lambda=-\Sigma+\omega\|<\infty$$

i.e,

$$(I-M_\lambda H)^{-1}G_\lambda\mathcal{B}_2(B)\subset\{\varphi\in\mathcal{Y}_2\,;\,\|\varphi\|\leqslant M\}.$$

Note that this last set is a bounded subset of $\mathcal{Y}_2$ which is independent of λ. The compactness of K together with (7.2) ensure then that

$$\lim_{|\mathrm{Im}\lambda|\to\infty}\sup_{\varphi\in\mathcal{Y}_2;\|\varphi\|\leqslant M}\|\mathcal{B}_1\Xi_\lambda K\varphi\|=0$$

which is the desired result.

Step 3: Let us show now that, for any $n\in\mathbb{N}$

$$\lim_{|\mathrm{Im}\lambda|\to\infty}\|\mathcal{B}_1\Xi_\lambda(M_\lambda H)^nG_\lambda\mathcal{B}_2\|=0\qquad\forall\,\mathrm{Re}\lambda=-\underline{\sigma}+\omega,\omega>0.\tag{7.3}$$

One writes $(M_\lambda H)^n=\sum_{j=1}^{2^n}P_j(\lambda)$ where $P_j(\lambda)$ is the product of n factors formed with $M_\lambda J$ and $M_\lambda K$ and where $P_{2^n}(\lambda)=(M_\lambda J)^n$. For $j\in\{1,\ldots,2^n-1\}$, the operator K appears at least once in the expression of $P_j(\lambda)$.

Step 3.1 : Let us prove that, for any $j\in\{1,\ldots,2^n-1\}$,

$$\lim_{|\mathrm{Im}\lambda|\to\infty}\|P_j(\lambda)G_\lambda\mathcal{B}_2\|_{\mathfrak{B}(\mathcal{X}_2,Y_2)}=0\qquad\forall\mathrm{Re}\lambda=-\Sigma+\omega,\omega>0.$$

The proof is once again inspired to that of Step 3.1 of Appendix 6. As above, it suffices to prove that, for any $k\geqslant 0$,

$$\lim_{|\mathrm{Im}\lambda|\to\infty}\|K(M_\lambda J)^kG_\lambda\mathcal{B}_2\|_{\mathfrak{B}(\mathcal{X}_2,Y_2)}=0\qquad\forall\mathrm{Re}\lambda>-\Sigma+\omega.\tag{7.4}$$

One may assume by a domination argument that the reflection coefficient $\gamma(\cdot)$ is constant and equals to one. Then, direct computations show that, for any $y\in[0,R]$, $K(M_\lambda J)^kG_\lambda\mathcal{B}_2\varphi(y)$ is equal to

$$\mathbf{g}(-\sqrt{R^2-y^2}/R)\int_0^R\mathbf{k}(\sqrt{R^2-\eta^2}/R)\exp\{-2k(\lambda+\Sigma)\sqrt{R^2-\eta^2}\}\frac{\eta}{R}\mathrm{d}\eta$$
$$\times\int_{-\sqrt{R^2-\eta^2}}^{\sqrt{R^2-\eta^2}}\alpha_2(\sqrt{z^2+\eta^2})\beta_2(z/\sqrt{z^2+\eta^2})\exp\{-(\lambda+\Sigma)(\sqrt{R^2-\eta^2}-z)\}\mathrm{d}z$$
$$\times\int_{-\sqrt{z^2+\eta^2}}^{\sqrt{z^2+\eta^2}}\theta_2(u/\sqrt{z^2+\eta^2})\varphi(u,\sqrt{z^2+\eta^2-u^2})\frac{\mathrm{d}u}{\sqrt{z^2+\eta^2}}.$$

Therefore, $K(M_\lambda J)^k G_\lambda \mathcal{B}_2$ splits as $K(M_\lambda J)^k G_\lambda \mathcal{B}_2 = \mathcal{R}_1(\lambda)\mathcal{R}_2$ where $\mathcal{R}_2 \in \mathfrak{B}(Y_2, Y_2)$ is given by

$$\mathcal{R}_2\varphi(\varrho) = \alpha_2(\varrho)\int_{-\varrho}^{\varrho} \theta_2(u/\varrho)\varphi(u, \sqrt{\varrho^2 - u^2})\frac{\mathrm{d}u}{\varrho} \qquad (\varrho \in [0, R])$$

and $\mathcal{R}_1(\lambda) \in \mathfrak{B}(Y_2, Y_2)$ given by

$$\begin{aligned}\mathcal{R}_1(\lambda)\psi(y) = \mathbf{g}(-\sqrt{R^2 - y^2}/R)\int_0^R \mathbf{k}(\sqrt{R^2-\eta^2}/R)\mathrm{d}\eta \\ \times \int_{-\sqrt{R^2-\eta^2}}^{\sqrt{R^2-\eta^2}} \beta_2(z/\sqrt{z^2+\eta^2})\psi(\sqrt{z^2+\eta^2}) \\ \exp\left\{-(\lambda+\Sigma)\left[(2k+1)\sqrt{R^2-\eta^2} - z\right]\right\}\mathrm{d}z.\end{aligned}$$

It is then enough to show that $\lim_{|\mathrm{Im}\lambda|\to\infty} \|\mathcal{R}_1(\lambda)\| = 0 \ \forall \mathrm{Re}\lambda = -\Sigma + \omega$. Splitting the last integral on integrals over $[-\sqrt{R^2-\eta^2}, 0[$ and $[0, \sqrt{R^2-\eta^2}[$ allows to write $\mathcal{R}_1(\lambda) = \mathcal{R}_1^-(\lambda) + \mathcal{R}_1^+(\lambda)$. It is then possible to perform the change of variables $z \mapsto \varrho = \sqrt{z^2+\eta^2}$ which shows that

$$\mathcal{R}_1^{\pm}(\lambda)\psi(y) = \mathbf{g}(-\sqrt{R^2-y^2}/R)\int_0^R \psi(\varrho)F_\lambda^{\pm}(\varrho)\varrho\mathrm{d}\varrho$$

where

$$\begin{aligned}F_\lambda^{\pm}(\varrho) = \pm\int_0^{\varrho} \mathbf{k}(\sqrt{R^2-\eta^2}/R)\beta_2(\sqrt{\varrho^2-\eta^2}/\varrho)\times \\ \times\exp\left\{-(\lambda+\Sigma)\left[(2k+1)\sqrt{R^2-\eta^2} \mp \sqrt{\varrho^2-\eta^2}\right]\right\}\frac{\mathrm{d}\eta}{\sqrt{\varrho^2-\eta^2}}\end{aligned}$$

Now, noting that $\omega(\eta) = (2k+1)\sqrt{R^2-\eta^2} \mp \sqrt{\varrho^2-\eta^2}$ fulfills the assumption of Lemma 2.6 for any ϱ, one has

$$\lim_{|\mathrm{Im}\lambda|\to\infty} F_\lambda^{\pm}(\varrho) = 0 \qquad \text{for a. e. } \varrho \in (0, R).$$

Moreover, one sees easily that $\int_0^R \sup_{\mathrm{Re}\lambda=-\Sigma+\omega} |F_\lambda^{\pm}(\varrho)|^2\varrho\,\mathrm{d}\varrho < \infty$ and the Dominated Convergence Theorem shows that

$$\lim_{|\mathrm{Im}\lambda|\to\infty}\int_0^R \left|F_\lambda^{\pm}(\varrho)\right|^2\varrho\,\mathrm{d}\varrho = 0 \qquad \forall \mathrm{Re}\lambda = -\Sigma+\omega.$$

Since $\|\mathcal{R}_1^{\pm}(\lambda)\| \leqslant \|\mathbf{g}(\cdot)\|_{L^2}\|F_\lambda^{\pm}\|_{Y_2}$ one gets the conclusion.

Step 3.2 : Let us show that

$$\lim_{|\mathrm{Im}\lambda|\to\infty} \|\mathcal{B}_1\Xi_\lambda(M_\lambda J)^n G_\lambda\mathcal{B}_2\| = 0 \qquad \forall\ \mathrm{Re}\lambda = -\Sigma+\omega, \omega > 0. \tag{7.5}$$

Tedious calculations show that $\mathcal{B}_1\Xi_\lambda(M_\lambda J)^n G_\lambda\mathcal{B}_2$ splits as $\mathcal{B}_1\Xi_\lambda(M_\lambda J)^n G_\lambda\mathcal{B}_2 = \mathcal{A}_3\mathcal{A}_2(\lambda)\mathcal{R}_2$ where

$\mathcal{R}_2$ has been defined in Step 3.1, $\mathcal{A}_2(\lambda) \in \mathfrak{B}(Y_2)$ given by:

$$\begin{aligned}\mathcal{A}_2(\lambda)\psi(\eta) &= \mathcal{A}_2^1(\lambda)\psi(\eta) + \mathcal{A}_2^2(\lambda)\psi(\eta)\\ &= \int_{-\eta}^{\eta} \theta_1(z/\eta)\frac{\mathrm{d}z}{\eta}\int_{-\sqrt{R^2+z^2-\eta^2}}^{0} \beta_2(\frac{u}{\sqrt{u^2+\eta^2-z^2}})\\ &\times \psi(\sqrt{u^2+\eta^2-z^2})e^{-(\lambda+\Sigma)\left[(2n+2)\sqrt{R^2+z^2-\eta^2}-u+z\right]}\mathrm{d}u,\\ &+ \int_{-\eta}^{\eta} \theta_1(z/\eta)\frac{\mathrm{d}z}{\eta}\int_{0}^{\sqrt{R^2+z^2-\eta^2}} \beta_2(\frac{u}{\sqrt{u^2+\eta^2-z^2}})\\ &\times \psi(\sqrt{u^2+\eta^2-z^2})e^{-(\lambda+\Sigma)\left[(2n+2)\sqrt{R^2+z^2-\eta^2}-u+z\right]}\mathrm{d}u,\end{aligned}$$

and $\mathcal{A}_3\varphi(x,y) = \alpha_1(\sqrt{x^2+y^2})\beta_1(x/\sqrt{x^2+y^2})\varphi(\sqrt{x^2+y^2}) \in \mathcal{X}_2$, $\forall \varphi \in Y_2$. Therefore, it suffices to show that $\lim_{|\mathrm{Im}\lambda|\to\infty}\|\mathcal{A}_2(\lambda)\| = 0$ for any $\mathrm{Re}\lambda = -\Sigma+\omega$. Performing the change of variables $u \mapsto \varrho = \sqrt{u^2-z^2+\eta^2}$, one sees that $\mathcal{A}_2^2(\lambda) = \mathcal{I}_1(\lambda)+\mathcal{I}_2(\lambda)+\mathcal{I}_3(\lambda)$ where

$$\mathcal{I}_i(\lambda) = \int_0^R F_\lambda^i(\eta,\varrho)\psi(\varrho)\,\varrho\,\mathrm{d}\varrho \qquad (i=1,2,3),$$

with

$$F_\lambda^i(\eta,\varrho) = \int_{-\eta}^{\eta} g_i(\eta,\varrho,z)e^{-(\lambda+\Sigma)\left[(2n+2)\sqrt{R^2+z^2-\eta^2}-\sqrt{\varrho^2+z^2-\eta^2}+z\right]}\theta_1(z/\eta)\times \\ \times \frac{\beta_2(\sqrt{\varrho^2+z^2-\eta^2}/\varrho)}{\sqrt{\varrho^2+z^2-\eta^2}}\frac{\mathrm{d}z}{\eta},$$

where $g_1(\eta,\varrho,z) = \chi_{]0,\eta[}(\varrho)\chi_{]-\eta,-\sqrt{\eta^2-\varrho^2}[}(z)$, $g_2(\eta,\varrho,z) = \chi_{]0,\eta[}(\varrho)\chi_{]\sqrt{\eta^2-\varrho^2},\eta[}(z)$ and $g_3(\eta,\varrho,z) = \chi_{]\eta,R[}(\varrho)\chi_{]-\eta,\eta[}(z)$.

As in the Step 3.1, one notes that, according to the Riemann–Lebesgue Lemma 2.6, for a. e. $(\eta,\varrho) \in (0,R)\times(0,R)$, $\lim_{|\mathrm{Im}\lambda|\to\infty} F_\lambda^i(\eta,\varrho) = 0$ and, since

$$\int_0^R \eta\,\mathrm{d}\eta\int_0^R \sup_{\mathrm{Re}\lambda=-\Sigma+\omega}\left|F_\lambda^i(\eta,\varrho)\right|^2\varrho\,\mathrm{d}\varrho < \infty,$$

the Dominated Convergence Theorem implies

$$\lim_{|\mathrm{Im}\lambda|\to\infty}\int_0^R \eta\,\mathrm{d}\eta\int_0^R \left|F_\lambda^i(\eta,\varrho)\right|^2\varrho\,\mathrm{d}\varrho = 0, \qquad (\mathrm{Re}\lambda = -\Sigma+\omega,\ i=1,2,3).$$

We conclude by noting that $\|\mathcal{I}_i(\lambda)\| \leqslant \|F_\lambda^i(\cdot,\cdot)\|_{Y_2\times Y_2}$. This proves that $\lim_{|\mathrm{Im}\lambda|\to\infty}\|\mathcal{A}_2^2(\lambda)\| = 0$ for any $\mathrm{Re}\lambda > -\Sigma$. One proceeds in the same way for $\mathcal{A}_2^1(\lambda)$. The proof of Proposition 4.6 follows then by compiling all the above steps as in the proof of Proposition 3.6.

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